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Universal stability of hydromagnetic flows

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The non-linear stability of hydromagnetic flows is investigated by applying energy methods. A universal stability estimate, namely a stability limit for motions subject to arbitrary non-linear disturbances, is obtained for bounded or periodic domains. Our analysis is restricted to fluids possessing constant density and electrical conductivity and we do not take into account temperature or Hall effects. This result establishes the existence of an open region of certain stability near the origin of the $(\tilde{R}_e, \tilde{R}_m)$ Cartesian plane for every fixed P_m (where $\tilde{R}_e$, $\tilde{R}_m$ and P_m are the Reynolds number, magnetic Reynolds number and magnetic Prandtl number, respectively). The universal stability limit can then be improved by suitably defining a maximum problem using variational techniques, and obtaining the relevant Euler–Lagrange equations. The tentative solution to this problem gives a stability limit which enlarges the universal stability region. Our results are then compared with linear and experimental ones, with special emphasis given to the role played by the magnetic field.

1. Introduction

The linear stability of hydromagnetic flows has been investigated by many authors.† In the linear theory, disturbances are assumed infinitesimal and in many cases $\tilde{R}_m$, the magnetic Reynolds number, is assumed small while $\tilde{R}_e$, the Reynolds number, is varied. The neutral curves obtained in this manner indicate an increase in stability due to the presence of the magnetic field (namely, an increase in the critical $\tilde{R}_e$ when $\tilde{R}_m$, or the Alfvén number A, is increased). As in the case of non-magnetic flows, the linear stability limits which delineate criteria for instability are well above the critical parameters obtained by experiments.

† The linear stability of channel flows with aligned and crossed magnetic fields was investigated by Stuart (1954) and Lock (1955), respectively. More recently these problems were investigated by Tarasov (1960), Hains (1965) and Ko (1968). Hydromagnetic stability of boundary layers was analysed by Rossow (1958) and Abas (1968) while rotational Couette flow was treated by Chang & Sartory (1967). Experimental investigations were carried out by Murgatroyd (1953) and Donaldson (1962), for example. Despite the difficulties in conducting experiments with liquid metals, there is a renewed interest in experiments with liquid metal MHD generators (see Cerini & Elliot 1968). Such experiments can potentially provide pertinent stability information.

Other very important kinds of MHD instabilities due to Hall effects, which are not included in the present work, have been investigated theoretically and experimentally by Kerrebrock & Dethlefsen (1968) (electrothermal instability in non-equilibrium ionization where σ is a function of T and p), Velikhov (1962), McCune (1964) (magneto-acoustic waves interacting with Hall currents) and others.

Another factor emphasized lately by Hunt (1966) is the important role played by three-dimensional disturbances, as opposed to the two-dimensional ones normally considered by the linear theory (Squires' (1933), theorem), in the explanation of transition phenomena.

Recent non-linear stability investigations of non-magnetic flows (see Joseph & Carmi 1969) were successful in predicting the correct experimental instability mode for three-dimensional finite-amplitude disturbances. The disturbance propagating in a direction transverse to the main flow direction was found to be the most destabilizing, in agreement with the experimental results (e.g. Fox, Lessen & Bhat (1968) who investigated the stability of Hagen–Poiseuille flow).

As the linear theory can only predict instability and is unable to provide a stability criteria which can be tested experimentally, it is of great interest to apply the non-linear stability theory to the present problem. In this paper we will establish a universal stability estimate for arbitrary non-linear disturbances, in bounded or periodic domains, for hydromagnetic flows with constant density ρ and electrical conductivity σ, neglecting temperature and Hall effects. The result will then be improved using variational techniques. The energy method to be employed was described by Serrin (1959) and Joseph (1966) for viscous incompressible and thermoconvective flows. The estimates obtained will provide sufficient conditions for asymptotic stability and will establish regions of certain stability near the origin of the $(\tilde{R}_e, \tilde{R}_m)$ Cartesian plane. It should be noted that although our treatment considers the stability of a fluid with constant ρ and σ the analysis can be expanded to include slight compressibility effects. Temperature effects were recently included by Lalas & Carmi (1970*a*) in the energy stability investigation of a conducting Boussinesq fluid (where we assume density variations only appear in the buoyancy term and where the electrical and thermal conductivities were taken as constants). A similar analysis was also carried out for the thermoconvection of ferrofluids (see Lalas & Carmi 1970*b*).

2. Difference motion equations and energy identities

The governing equations of motion for a viscous fluid with constant density and finite conductivity flowing in a magnetic field are (see Chandrasekhar 1961)

$$\left.\begin{aligned} \frac{d\mathbf{V}}{dt} &= \frac{1}{\rho\mu_0}\mathbf{B}.\nabla\mathbf{B} - \frac{1}{\rho}\nabla\left(p + \frac{1}{2\mu_0}|B|^2\right) + \nu\nabla^2\mathbf{V}, \\ \frac{d\mathbf{B}}{dt} &= \mathbf{B}.\nabla\mathbf{V} + \frac{1}{\sigma\mu_0}\nabla^2\mathbf{B}, \\ \nabla.\mathbf{V} &= 0, \quad \nabla.\mathbf{B} = 0, \end{aligned}\right\} \qquad (1a,b,c)$$

where $\mathbf{V}$ = velocity vector, $\mathbf{B}$ = magnetic flux density vector, ρ = density o fluid, p = pressure, μ_0 = magnetic permeability, ν = kinematic viscosity and σ = electrical conductivity.

We will consider bounded domains $\mathscr{V}$ with pre-assigned boundary conditions $\mathbf{V}$ and $\mathbf{B}$ on the rigid surface $\partial\mathscr{V}$ of $\mathscr{V}$.

To analyse the stability of the basic motion $(\mathbf{V}, \mathbf{B}, p)$ we consider an altered motion ($\mathbf{V}^*, \mathbf{B}^*, p^*$) satisfying the same equations (1) and the same boundary conditions, but differing from this state initially. By subtracting the relevant governing equations we obtain the difference motion equations (see Joseph 1966)

$$\left.\begin{aligned} \frac{d\mathbf{u}}{dt}+\mathbf{u}.\nabla\mathbf{V}+\mathbf{u}.\nabla\mathbf{u} &= \frac{1}{\rho\mu_0}(\mathbf{b}.\nabla\mathbf{B}+\mathbf{b}.\nabla\mathbf{b}+\mathbf{B}.\nabla\mathbf{b}) \\ &\quad -\frac{1}{\rho}\nabla\left[p^*-p+\frac{1}{2\mu_0}(|B^*|^2-|B|^2)\right]+\nu\nabla^2\mathbf{u}, \\ \frac{d\mathbf{b}}{dt}+\mathbf{u}.\nabla\mathbf{B}+\mathbf{u}.\nabla\mathbf{b} &= \mathbf{B}.\nabla\mathbf{u}+\mathbf{b}.\nabla\mathbf{u}+\mathbf{b}.\nabla\mathbf{V}+\frac{1}{\sigma\mu_0}\nabla^2\mathbf{b}, \\ \nabla.\mathbf{u} = 0, \quad \nabla.\mathbf{b} &= 0 \quad \text{in} \quad \mathscr{V}, \end{aligned}\right\} \qquad (2a, b, c)$$

with homogeneous boundary conditions

$$\mathbf{u} = \mathbf{b} = 0 \quad \text{on} \quad \partial\mathscr{V}, \qquad (3)$$

and where we defined the difference variables $\mathbf{u}$, $\mathbf{b}$ as

$$\mathbf{u} = \mathbf{V}^*-\mathbf{V} \quad \text{and} \quad \mathbf{b} = \mathbf{B}^*-\mathbf{B}.$$

We have to determine now the conditions under which the altered flow will tend asymptotically to the basic flow as $t \to \infty$.

To do that we define $\check{K}_1$ and $\check{K}_2$, the kinetic and magnetic energies, respectively,

$$\check{K}_1 = \int_{\mathscr{V}} \tfrac{1}{2}u^2 d\mathscr{V}, \quad \check{K}_2 = \int_{\mathscr{V}} \tfrac{1}{2}b^2 d\mathscr{V}. \qquad (4)$$

We say that the basic motion is asymptotically stable in the mean if $\check{K}_1 \to 0$ and $\check{K}_2 \to 0$ as $t \to \infty$. The rates of change of $\check{K}_1$ and $\check{K}_2$ are governed by

$$\frac{d\check{K}_1}{dt} = -\int_{\mathscr{V}}(\mathbf{u}.\mathsf{D}.\mathbf{u}+\nu\nabla\mathbf{u}:\nabla\mathbf{u})+\frac{1}{\rho\mu_0}\int_{\mathscr{V}}(\mathbf{B}.\nabla\mathbf{b}.\mathbf{u}+\mathbf{b}.\nabla\mathbf{b}.\mathbf{u}+\mathbf{b}.\nabla\mathbf{B}.\mathbf{u}), \qquad (5a)$$

$$\frac{d\check{K}_2}{dt} = -\int_{\mathscr{V}}\left(-\mathbf{b}.\mathsf{D}.\mathbf{b}+\frac{1}{\sigma\mu_0}\nabla\mathbf{b}:\nabla\mathbf{b}\right)+\int_{\mathscr{V}}(\mathbf{B}.\nabla\mathbf{u}.\mathbf{b}+\mathbf{b}.\nabla\mathbf{u}.\mathbf{b}-\mathbf{u}.\nabla\mathbf{B}.\mathbf{b}), \qquad (5b)$$

which in turn are obtained by integrating the following equations

$$\frac{d\frac{1}{2}u^2}{dt} = -\mathbf{u}.\mathsf{D}.\mathbf{u}-\nu\nabla\mathbf{u}:\nabla\mathbf{u}+\frac{1}{\rho\mu_0}(\mathbf{B}.\nabla\mathbf{b}.\mathbf{u}+\mathbf{b}.\nabla\mathbf{b}.\mathbf{u}+\mathbf{b}.\nabla\mathbf{B}.\mathbf{u})+\nabla.\mathbf{A}_1, \qquad (6a)$$

$$\frac{d\frac{1}{2}b^2}{dt} = +\mathbf{b}.\mathsf{D}.\mathbf{b}-\frac{1}{\sigma\mu_0}\nabla\mathbf{b}:\nabla\mathbf{b}+(\mathbf{B}.\nabla\mathbf{u}.\mathbf{b}+\mathbf{b}.\nabla\mathbf{u}.\mathbf{b}-\mathbf{u}.\nabla\mathbf{B}.\mathbf{b})+\nabla.\mathbf{A}_2, \qquad (6b)$$

and using the divergence constraints ($2c$) and the boundary conditions (3). Here D is the strain rate tensor of the basic motion and

$$\left.\begin{aligned} \mathbf{A}_1 &= \nu\nabla\tfrac{1}{2}u^2-\mathbf{u}\left[\tfrac{1}{2}u^2+\frac{p^*-p}{\rho}+\frac{1}{2\mu_0\rho}(|B^*|^2-|B|^2)\right], \\ \mathbf{A}_2 &= \frac{1}{\sigma\mu_0}\nabla\tfrac{1}{2}b^2-\mathbf{u}\tfrac{1}{2}b^2. \end{aligned}\right\} \qquad (7a, b)$$

Equations ($6a, b$) are obtained by dotting $\mathbf{u}$, $\mathbf{b}$ into ($2a$), ($2b$), respectively.

The energy functionals (5*a*) and (5*b*) also hold for unbounded domains $\mathscr{V}$ when the flow geometry is such that the disturbances can be assumed spatially periodic at each instant.

The dimensionless form of (5) is

$$\frac{dK_1}{d\tau} = -\int_{\mathscr{V}} (R_e \mathbf{v}.\mathsf{E}.\mathbf{v} + \nabla\mathbf{v}\colon \nabla\mathbf{v} - R_m \mathbf{h}.\nabla\mathbf{H}.\mathbf{v}) + C, \tag{8a}$$

$$P_m \frac{dK_2}{d\tau} = -\int_{\mathscr{V}} (-P_m R_e \mathbf{h}.\mathsf{E}.\mathbf{h} + \nabla\mathbf{h}\colon \nabla\mathbf{h} + R_m \mathbf{v}.\nabla\mathbf{H}.\mathbf{h}) - C, \tag{8b}$$

where $\mathbf{x} = \check{\mathbf{x}}/d$, d = characteristic length, $\tau = (\nu/d^2)\,t$, $\mathbf{v} = \mathbf{u}/U_0$, U_0 = reference velocity (typically the maximum velocity in $\mathscr{V}$), $\mathsf{E} = \mathsf{D}/m$, $-m$ = least characteristic value of D, $P_m = \mu_0\sigma\nu$ = magnetic Prandtl number, $\mathbf{h} = \mathbf{b}A/B_0P_m^{\frac{1}{2}}$, B_0 = maximum magnetic field in $\mathscr{V}$, $A = B_0/U_0(\rho\mu_0)^{\frac{1}{2}}$ = Alfvén number, $\mathbf{H} = \mathbf{B}A/B_0P_m^{\frac{1}{2}}$, $R_e = md^2/\nu$ = Reynolds number and $R_m = \sigma\mu_0 dU_0$ = magnetic Reynolds number. The magnetic Reynolds number R_m is the ratio of the convection rate to the diffusion rate of the magnetic field, while the Reynolds number R_e is a similar ratio for the vorticity field. Large R_m implies a thin boundary layer in which dissipation occurs. Outside this region the magnetic field and the flow are 'frozen' together. Small R_m on the other hand, implies that the total magnetic field in the flow is essentially equal to the imposed one, so that the induced field is small. The magnetic Prandtl number P_m gives an indication of the relative diffusion of vorticity to the diffusion of the magnetic field.

In the above

$$K_1 = \int_{\mathscr{V}} \tfrac{1}{2}v^2, \quad K_2 = \int_{\mathscr{V}} \tfrac{1}{2}b^2$$

and

$$\begin{aligned} C &= R_m \int_{\mathscr{V}} (\mathbf{H}.\nabla\mathbf{h}.\mathbf{v} + \mathbf{h}.\nabla\mathbf{h}.\mathbf{v}) \\ &= -R_m \int_{\mathscr{V}} (\mathbf{H}.\nabla\mathbf{v}.\mathbf{h} + \mathbf{h}.\nabla\mathbf{v}.\mathbf{h}). \end{aligned}$$

The last relation follows after integration by parts and using the boundary conditions and divergence constraints. Note that the above definitions of $\mathbf{h}$ and $\mathbf{H}$ suppress the dimensionless parameter A. In the following analysis only the energy identities and kinematic constraints will be utilized, with no further use of the local non-linear conservation equations.

Adding (8*a*) and (8*b*) gives the rate of change of the total energy E

$$\begin{aligned} \frac{dE}{d\tau} = \frac{d}{d\tau}[K_1 + P_m K_2] = -\int_{\mathscr{V}} [&\nabla\mathbf{v}\colon \nabla\mathbf{v} + \nabla\mathbf{h}\colon \nabla\mathbf{h} \\ &+ R_e \mathbf{v}.\mathsf{E}.\mathbf{v} - R_e P_m \mathbf{h}.\mathsf{E}.\mathbf{h} + 2R_m \mathbf{v}.\mathsf{G}.\mathbf{h}], \end{aligned} \tag{9}$$

where G = antisymmetric part of $\nabla\mathbf{H}$.

3. Universal stability estimate

In this section we generalize the stability criteria developed by Serrin (1959) and Joseph (1966) to hydromagnetic flows. According to these criteria, if the right-hand side of (9) is negative, for an arbitrary class of functions $\mathbf{v}$, $\mathbf{h}$ satisfying

$\nabla . \mathbf{v} = \nabla . \mathbf{h} = 0$ and homogeneous boundary conditions, we have asymptotic stability in the mean. The term

$$-\int_{\mathscr{V}} (\nabla \mathbf{v} \colon \nabla \mathbf{v} + \nabla \mathbf{h} \colon \nabla \mathbf{h})$$

is always negative; hence viscous and magnetic dissipation always stabilize the flow. The remaining three integrals in the right-hand side of (9) do not have definite signs and therefore can potentially destabilize the flow for critical values of the parameters R_e, R_m (P_m will be kept fixed in the following). We will first prove the existence of a neighbourhood near the origin of the (R_e, R_m) plane in which the motion is certainly stable.

THEOREM. *Let $\mathscr{V} = \mathscr{V}(t)$ be a bounded or periodic region of space and* $\mathbf{V}$, $\mathbf{B}$ *the velocity and the magnetic flux density vectors, respectively, satisfying prescribed conditions on $\partial\mathscr{V}$. Then $E = (K_1 + P_m K_2)$ satisfies*

$$E \leqslant E_0 \exp(-2MN\tau), \tag{10}$$

where $M = \delta - (\tilde{R}_e + \tilde{R}_m)$, $N = \min(1, 1/P_m)$, $\tilde{R}_e = \max(R_e, R_e P_m)$, $\tilde{R}_m = nR_m$ and $n = \max|\mathsf{G}|$.

Here $E_0 = K_{10} + P_m K_{20}$ is the initial disturbance energy and δ is defined in (12) *below. If $M > 0$ for all τ, then $E \to 0$ as $t \to \infty$ and the flow is asymptotically stable in the mean.*

We consider flows for which n exists in the closure of $\mathscr{V}$.

Proof. By the Schwarz inequality

$$\left.\begin{aligned} \left|\int_{\mathscr{V}} \mathbf{v}.\mathsf{E}.\mathbf{v}\right| &\leqslant \int_{\mathscr{V}} |\mathbf{v}|\,|\mathsf{E}|\,|\mathbf{v}| \leqslant 2K_1, \\ \left|\int_{\mathscr{V}} \mathbf{h}.\mathsf{E}.\mathbf{h}\right| &\leqslant \int_{\mathscr{V}} |\mathbf{h}|\,|\mathsf{E}|\,|\mathbf{h}| \leqslant 2K_2, \\ \left|\int_{\mathscr{V}} 2\mathbf{v}.\mathsf{G}.\mathbf{h}\right| &\leqslant 2\int_{\mathscr{V}} |\mathbf{v}|\,|\mathsf{G}|\,|\mathbf{h}| \leqslant 4n(K_1K_2)^{\frac{1}{2}} \leqslant 2n(K_1+K_2). \end{aligned}\right\} \tag{11}$$

For $\mathbf{v} = \mathbf{h} = 0$ on $\partial\mathscr{V}$ and $\operatorname{div}\mathbf{v} = \operatorname{div}\mathbf{h} = 0$ in $\mathscr{V}$ we have

$$\left.\begin{aligned} \int_{\mathscr{V}} \nabla\mathbf{v} \colon \nabla\mathbf{v} \geqslant 2\delta K_1, \\ \int_{\mathscr{V}} \nabla\mathbf{h} \colon \nabla\mathbf{h} \geqslant 2\delta K_2, \end{aligned}\right\} \tag{12}$$

where δ varies for various flow geometries as described in Shir & Joseph (1968). For spherical regions $\delta \simeq 80$ is the least positive root of $\tan(\frac{1}{2}\delta)^{\frac{1}{2}} = (\frac{1}{2}\delta)^{\frac{1}{2}}$ (see Payne & Weinberger 1961) and for other geometries the exact bounds were given by Velte (1962) and Sorger (1966) (e.g. for channel flow $\delta = 3{\cdot}74\pi^2$).

Using (11), (12) in (9) gives

$$dE/d\tau \leqslant -2N[\delta - (\tilde{R}_e + \tilde{R}_m)]\,E, \tag{13}$$

from which (10) follows upon integration, proving the theorem.

For stability we must have the condition

$$\tilde{R}_e + \tilde{R}_m \leqslant \delta. \tag{14}$$

This relation is plotted in figure 1 and bounds region A of universal stability. It should be noted that the magnetic field appears implicitly through m in $\tilde{R}_e$, while the magnetic field gradient appears explicitly through $\tilde{R}_m$.

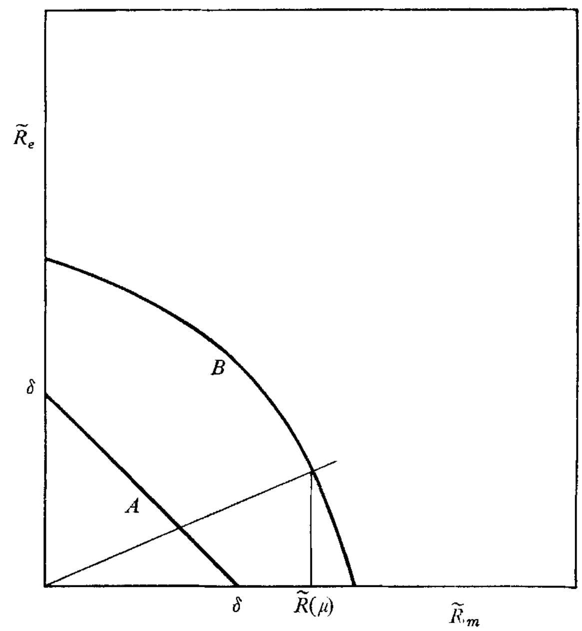

FIGURE 1. Universal and improved stability regions. A, universal stability region bounded by $\tilde{R}_e + \tilde{R}_m < \delta$; B, improved stability region bounded by $\tilde{R}(\mu)$ where $\mu = R_e/R_m$.

When the conductivity is zero, $\tilde{R}_m = 0$ and we recover Serrin's (1959) stability result $R_e \leqslant \delta$, for non-conducting flows. For stationary fluids, one has to further specify U_0. Since the dominant mechanism of propagation of a disturbance in a constant density fluid at rest with a magnetic field is the Alfvén speed, we take $U_0 = B_{\max}/(\mu_0 \rho)^{\frac{1}{2}}$ giving $R_m = \sigma d B_{\max} (\mu_0/\rho)^{\frac{1}{2}}$. In this case the magnetic field appears explicitly in the stability criteria $\tilde{R}_m \leqslant \delta$.

Using (14) we can also prove the uniqueness of steady flows in bounded regions.

COROLLARY. *Let* $\mathbf{V}^*$, $\mathbf{B}^*$, p^* *and* $\mathbf{V}$, $\mathbf{B}$, p *be the velocity, magnetic flux density and pressure, respectively, of two steady flows in* $\mathscr{V}$, *subject to the same boundary conditions on* $\partial\mathscr{V}$. *Then the two flows are identical, provided* (14) *holds.*

Proof. Since the flow is steady, K_1 and K_2 must be constant. On the other hand they must satisfy (13). As (14) is also satisfied, we must have

$$K_1 = K_{1_0} = K_2 = K_{2_0} = 0,$$

implying $\mathbf{V}^* = \mathbf{V}$, $\mathbf{H}^* = \mathbf{H}$, $p^* = p$ (almost everywhere), proving the corollary.

4. Improved stability region

The result obtained in the previous section is rigorous but conservative because of the rather crude estimates of the right-hand side of (9). An improved stability region is obtained by applying calculus of variation techniques to the above problem. For this purpose, we rewrite (9) as follows

$$\frac{dE}{d\tau} = \mathscr{D}\left[-1+R_m\left(\frac{-I}{\mathscr{D}}\right)\right] \leqslant \mathscr{D}\left(-1+\frac{R_m}{R}\right), \tag{15}$$

with
$$\mathscr{D} = \int_{\mathscr{V}} (\nabla \mathbf{v} \colon \nabla \mathbf{v} + \nabla \mathbf{h} \colon \nabla \mathbf{h}),$$

$$I = \int_{\mathscr{V}} (\mu \mathbf{v}.\mathsf{E}.\mathbf{v} - \mu P_m \mathbf{h}.\mathsf{E}.\mathbf{h} + 2\mathbf{v}.\mathsf{G}.\mathbf{h}),$$

$$\mu = R_e/R_m,$$

and where
$$\frac{1}{R} = \max_{\mathbf{v},\mathbf{h}}\left(\frac{-I}{\mathscr{D}}\right). \tag{16}$$

Solving the maximum problem (16) for all admissible functions $\mathbf{v}$, $\mathbf{h}$ which satisfy the divergence constraint and homogeneous boundary conditions will render an improved stability criteria. We will have certain stability if

$$\tilde{R}_m \leqslant \tilde{R}(\mu), \quad \text{where} \quad \tilde{R}(\mu) = nR(\mu).$$

The extremum problem $\max(-I)$ subject to the additional normalizing constraint $\mathscr{D} = 1$, leads to the following Euler–Lagrange equations,

$$\left.\begin{aligned} \mu R\mathbf{v}.\mathsf{E} + R\mathbf{h}.\mathsf{G} &= -\nabla p + \nabla^2 \mathbf{v}, \\ -\mu P_m R\mathbf{h}.\mathsf{E} + R\mathbf{v}.\mathsf{G} &= -\nabla p_1 + \nabla^2 \mathbf{h}, \end{aligned}\right\} \tag{17a, b}$$

where $2p/R$, $2p_1/R$ were introduced as Lagrange multipliers of the divergence constraints. The least positive eigenvalue of (17) for each μ (with P_m fixed) will give a limit stability point in the $(\tilde{R}_e, \tilde{R}_m)$ plane. Solving for the various μ will give us a curve in the parameter plane $(\tilde{R}_e, \tilde{R}_m)$ delineating an improved stability region. This curve is tentatively shown in figure 1 bounding region B. The above eigenvalue problem can be solved for various geometries and physical situations and the stability limit obtained can be compared with experimental and linear theory results. It can also provide a valuable guide in the design of future stability experiments especially in the prediction of critical modes (as shown for other problems by Joseph & Carmi 1969).

5. Discussion

The value of the linear theory lies in its ability to predict instability. On the other hand, by the energy theory, we can obtain a criterion for certain stability. In this sense the two theories are complementary. In this paper we discussed the energy criteria of stability of hydromagnetic flows and first proved rigorously the existence of a stable region near the origin of the $(\tilde{R}_e, \tilde{R}_m)$ plane (P_m = constant). The functions admitted in the search of our estimates do not necessarily satisfy the conservation of momentum and energy equations. They are kinematically admissible functions (i.e. satisfy continuity and the boundary conditions), but may be dynamically inadmissible.

The estimate obtained is very conservative as it ensures stability of the flow subject to all disturbances, even to those that violate conservation requirements. Note that the stability limit is reduced by increasing either one of the parameters, as opposed to linear theory results where the introduction of a magnetic field inhibited instability.

Next we applied variational techniques to improve our estimates and tentatively obtain an optimum stability limit. The actual solution of the derived Euler–Lagrange equations for various physical situations should now follow. Results obtained for non-magnetic flows are able to predict the correct critical disturbance mode for non-linear disturbances of finite amplitude. As mentioned in the introduction, for Hagen–Poiseuille flows, the energy results (Joseph & Carmi 1969) predicted that the azimuthal wave disturbance plays a dominant part in transferring energy from the basic to the difference motion as was also demonstrated by experiments (Fox *et al.* 1968). In this respect the energy theory gives a result which is more useful than the one obtained by the linear theory. We believe that similar results will hold for hydromagnetic flows but further experimental verification will be necessary.

There also exist flows (see Joseph 1966) where no subcritical instabilities are possible. By subcritical instabilities we mean those instabilities which occur at parameter values lower than the one given as critical by the linear theory. For such cases the experimental results coincide with the necessary and sufficient conditions for stability provided by the linear and energy theories, respectively. Subsequent investigations should fully analyse the role played by the magnetic field on the stability of hydromagnetic flows and in changing the range where subcritical instabilities are possible. Finally, we would like to point out the possible application of the energy theory techniques to two- and three-fluid models, which are of great practical importance in MHD generator design.

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